UTM ANNUAL TEACHER WORKSHOP

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GEOMETRIC

PROBABILITY

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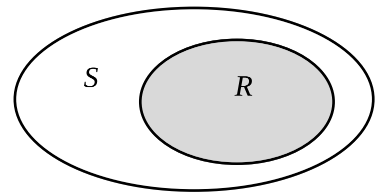
Definition: *If a point x is chosen at random within a certain region S then the probability that the point x lies within a sub–region R is equal to*

$$P\left(x\in R\right) := \frac{measure\left(R\right)}{measure\left(S\right)}$$



Notes: *The definition is based on the assumption that the probabilities are distributed uniformly.*

The "measures" of the regions must be the same: either both are lengths, or both areas, or both volumes.

Example 1:

Each of two professors comes to a parking lot at the university at a random time between 5:00pm and 6:00pm. One day, the professors agreed to wait for 15 minutes at a parking lot and only then go home. Find the probability that the professors meet.

Solution:

Let us denote the professors as *X* and *Y* and their arrival times as *x* and *y* respectively

We can assume that *x* and *y* are uniformly distributed between 0 and 1: $0\le x\le 1$ and $0\le y\le 1$.

The sample space of the problem will be represented by the set of points $\left(x,y\right)$ located within the unit square in the Cartesian coordinate system.

We will to consider two mutually exclusive cases: $y > x$ and $y\le x$.

Case $y > x$: (*Y arrives later than X*)

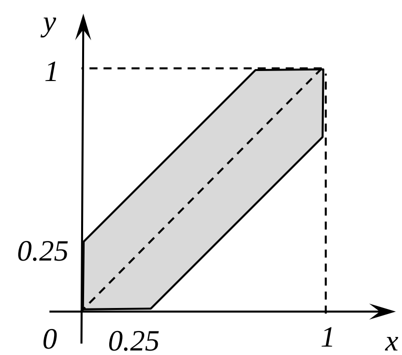
X and *Y* will meet if *Y* arrives not later than 15 minutes after *X*.

In other words, *x* and *y* must satisfy the inequality $y\le x+0.25$

Case $y\le x$: (*Y arrives not later than X*)

X and *Y* will meet if *Y* arrives not earlier than 15 minutes before *X*.

In other words, *x* and *y* must satisfy the inequality $y\ge x-0.25$



Both cases can be combined in a system of inequalities: $\begin{cases} 0\le x\le 1 \\ 0\le y\le 1 \\ y\le x+0.25 \\ y\ge x-0.25 \end{cases}$

All points $\left(x,y\right)$ that satisfy these conditions lie within the shaded region shown in the diagram above.

The area of that region is equal to $2\cdot\left(\frac{1}{2}-\frac{1}{2}\cdot\left(1-\frac{1}{4}\right)^{2}\right)=2\cdot\left(\frac{1}{2}-\frac{1}{2}\cdot\frac{9}{16}\right)=\frac{7}{16}$

Since the area of the unit square is equal to 1, the sought–for probability is equal to $\frac{7}{16}$

Answer: $\frac{7}{16}$

Example 2:

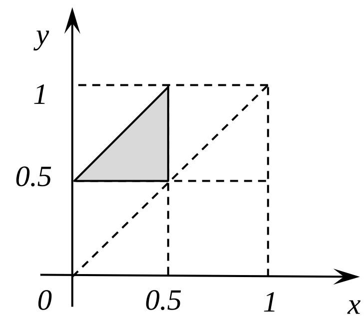
A line segment is broken into 3 parts by 2 points chosen at random. Find the probability that the obtained line segments can be used to construct a triangle.

Solution:

Let us assume that the length of the original line segment is equal to 1 and denote the coordinates of 2 points chosen at random as x and y. The sample space of our experiment will consists of all ordered pairs (x, y) that satisfy the conditions $0 \le x \le 1$ and $0 \le y \le 1$.

In a Cartesian coordinate system, this sample space can be represents by the square with the side length 1 bounded by the coordinate axes and lines $x = 1$, $y = 1$, and a pair (x, y) will be represented by a point randomly chosen within that square.

We will consider two mutually exclusive cases: $y > x$ and $y \le x$.



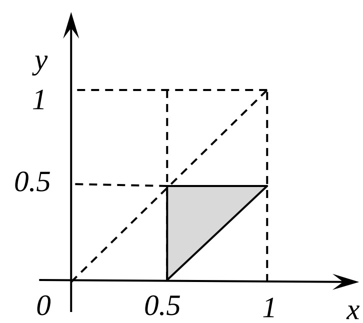
In the case $y > x$, the lengths of the obtained line segments are be equal to

$$x, \qquad (y - x), \qquad (1 - y).$$

The line segments can be used as the sides of a triangle if they satisfy the *triangle inequalities*:

$$\begin{cases} x < (y - x) + (1 - y) \\ (y - x) < x + (1 - y) \\ (1 - y) < x + (y - x) \end{cases} \Leftrightarrow \begin{cases} x < 0.5 \\ y < x + 0.5 \\ y > 0.5 \end{cases}$$

The points that satisfy $y > x$ and the system of inequalities shown above lies within the shaded triangle shown at the right (the top figure).



By analogy, the case $y \le x$ leads to shaded shown at the right (the bottom figure).

Therefore, the sought–for probability will be equal to

$$\frac{Area\ of\ the\ shaded\ triangles}{Area\ of\ the\ unit\ square} = \frac{0.25}{1} = 0.25$$

Answer: 0.25

Exercises:

1. A kitchen floor is covered with square tiles with side lengths 10 cm. A coin with a radius 2 cm is dropped on the floor randomly. Find the probability that the coin lies within one tile entirely.

2. An integer x is randomly chosen on a line segment [0, 100]. Find the probability that the integer part of the square root of the number x is even.

3. A real number x is randomly chosen on a line segment (0, 1000). Find the probability that the integer part of the cubed root of the number x is prime.

4. A point P is chosen at random on a line segment connecting points (0.1) and (3,4) on a coordinate grid. Find the probability that the area of the triangle with vertices at points (0,0) , (3,0), and P is greater than 2.

5. Two professors come to the parking lot at random times between 18:00 and 19:00. Find the probability that the professors meet if it takes one of them 5 minutes to find her car and the other find her car in 6 minutes.

6. Two people usually come to a specfied place independently from each other at a random times between 13:00 and 14:00, wait for each other for not more than 18 minutes and then leave. By how much, in percent, will the probability that they meet change if the waiting time is increased to 36 minutes?

7. Jane sends an e–mail message to Steve every day at a random time 14:00 and 16:00. Steve checks his e-mails every day at a random time between 15:00 and 17:00. Find the probability that Jane's message will be in Steve's mailbox by the time he opens the e–mail software. Assume that e–mails arrive instantly.

8. Two trains arrive to the same station at random times between noon and 1:00pm, independently from each other. The wait times of the trains are equal. Find the wait time of the trains, in minutes, given that the probability that one of the trains arrives before the other train leaves is equal to 0.64?

9. Two trains arrive to the same station at random times between noon and 2:00pm, independently from each other. The wait times of the trains are equal. Find the wait time of the trains, in minutes, given that the probability that one of the trains arrives before the other train leaves is equal to 0.19?

10. Find the probability that the length of a chord drawn in a circle exceeds the length of the radius of the circle if the endpoints of the chord are randomly chosen on the circumference of the circle.

11. Three real numbers x, y, z are chosen at random on the line segment [0,1]. Find the probability that $x \le y \le z$.

12. Three real numbers a, b, c are chosen at random on the line segment [0,1]. Find the probability that it is possible to construct a triangle with side lengths a, b, c.

13. Three points are chosen on the circumference of a circle at random. Find the probability that the center of the circle lies within the triangle with the vertices at those points.

14. Three points are chosen at random on the circumference of a circle, and the triangle with vertices at those points is constructed. Find the probability that none of the side lengths of the triangle is longer than the radius of the circle.

<u>*Answers*</u>: 1. 0.64; 2. 0.45; 3. 0.316; 4. $\frac{1}{9}$; 5. ≈0.175; 6. ≈65%; 7. $\frac{7}{8}$; 8. 24 min; 9. 12 min; 10. $\frac{2}{3}$; 11. $\frac{1}{6}$; 12. 0.5; 13. 0.25; 14. $\frac{1}{9}$;